Group Theory 11

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Simple Group :- A group $G \neq 1$ is simple if it has no normal subgroups except 1 and G.

∘> H is a maximal normal group of G iff G/H is simple

Theorem :- Let G be a group with normal subgroups H and K. If $HK = G$ and $H \cap K = 1$ then $G \cong H \times K$

Proof :- $hk \in G$ where $h \in H$ and $k \in K$

$h_1k_1, h_2k_2 \in G$ and $h_1 k_1 = h_2 k_2$

$$h_2^{-1} h_1 = k_2 k_1^{-1} \in H \cap K = 1$$

$$h_2^{-1} h_1 = 1 \Rightarrow h_2 = h_1$$

$$k_2 = k_1$$

So, hk is uniquely determined by some $h \in H$ and $k \in K$

Let us take,

$$f : HK \longrightarrow H \times K$$

$$hk \longrightarrow (h, k)$$

$f(h_1k_1) = (h_1, k_1)$

$f(h_2k_2) = (h_2, k_2)$

$f(h_1 \underbrace{k_1 h_2} k_2)$

$= f(h_1 h_2 k_1 k_2)$

$= f(h_3 k_3) = (h_3, k_3)$

$= (h_1 h_2, k_1 k_2)$

$= (h_1, k_1)(h_2, k_2)$

$= f(h_1k_1) f(h_2k_2)$

$(h k h^{-1}) k^{-1} \in K$

$h (k h^{-1} k^{-1}) \in H$

$h k h^{-1} k^{-1} \in H \cap K = 1$

$k h k^{-1} h^{-1} \in H \cap K = 1$

h and k commute

So f is a isomorphism.

So $HK \cong H \times K$, $G \cong H \times K$

<u>Theorem:-</u> If $A \lhd H$ and $B \lhd K$ then $A \times B \lhd H \times K$ and

$$(H \times K)/(A \times B) \cong (H/A) \times (K/B)$$

<u>Proof:-</u> $f \quad H \times K \longrightarrow (H/A) \times (K/B)$

$\quad (h, k) \longrightarrow (Ah, Bk)$

surjective $\qquad \ker(f) = A \times B$

$(H \times K)/(A \times B) \cong (H/A) \times (K/B)$ using 1st Isomorphism thm.

•> If $G = H \times K$ then $G/(H \times 1) \cong K$

Q> Prove that the abelian group G of order p^2 (p is prime) is either a cyclic or isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$

Ans:- $\mathbb{Z}/p^2$ for G cyclic

If G not cyclic then,

non-identity $x \in G$, $\langle x \rangle \qquad G \setminus \langle x \rangle$

$y \in G \setminus \langle x \rangle$

If $\quad y x_1 \in \langle x \rangle$ for $x_1 \in \langle x \rangle$

$y \in \langle x \rangle \quad \Rightarrow \Leftarrow$

So $y x_1 \in G \setminus \langle x \rangle$

If $yx_1y^{-1} \in G \setminus \langle x \rangle$

So $\langle x \rangle$ is normal $\rightarrow y \in G \setminus \langle x \rangle$

Similarly $\langle y \rangle$ is also normal

$\langle x \rangle \cap \langle y \rangle = 1$

$\text{Ord}(\langle x \rangle) \mid p^2 \Rightarrow \text{Ord}(\langle x \rangle) = 1, p, p^2$

$y \in G \setminus \langle x \rangle$

$\text{Ord}(\langle y \rangle) \mid p^2 \Rightarrow \text{Ord}(\langle y \rangle) = 1, p, p^2$

$\text{Ord}(\langle x \rangle) = p$

$\text{Ord}(\langle y \rangle) = p$ $\Rightarrow$ $\langle x \rangle \langle y \rangle = G$

So, $G \cong \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$

•> If p is a prime then an elementary abelian p-group is a finite group G isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p \times \dots \times \mathbb{Z}/p$

Q> Let H be a subgroup of a group G such that $[G:H] = 2$. Prove that H is a normal subgroup of G

Ans:- $a \in G \setminus H$. $[G:H] = 2$, H and aH are cosets

H and Ha are cosets

$aH = Ha$

$aHa^{-1} = H$

So H is normal

Q> Let a be an element of a group G such that $\text{Ord}(a)$ is finite. If H is normal subgroup of G, Q> ... If H is normal subgroup ... Ord(a) is finite. If H is normal subgroup, then prove that Ord(aH) divides Ord(a)

Ans:— $aHa^{-1} = H$ $\qquad n = Ord(a) \Rightarrow a^n = 1 \rightarrow$ smallest value n

$(aH)^n = aHaH \cdots aH$

$= aaHH \cdots aH$

$= a^n H^n = a^n H = eH = H$

So $Ord(aH) \mid n \Rightarrow Ord(aH) \mid Ord(a)$

Q> Let G be an Abelian group with an odd number of elements. Prove that the product of all elements of G is the identity

Ans:— $|G| = \{a_1, a_2, \ldots, a_{2n+1}\}$ $\qquad a_{2n+1} = e$

Suppose $a \in G$ and $a^2 = e$ and $a \neq e$

$\{e, a\}$ is subgroup

$\Rightarrow |\{e, a\}| \mid |G| \Rightarrow 2 \mid 2n+1 \Rightarrow \Leftarrow$ contradiction

thus $a^2 = e$ is not for any $a \in G$

So a and $a^{-1} \in G$ where $a \neq a^{-1}$

$a_1 a_1^{-1} \cdots a_n a_n^{-1} e = e$

Q> Prove that $\mathbb{Q}$ under addition is not isomorphic to $\mathbb{Q}^*$ under multiplication

Ans:— Suppose they are isomorphic by $f: \mathbb{Q} \rightarrow \mathbb{Q}^*$
$0 \rightarrow 1$

$p \in \mathbb{Q}^*$

$f(\alpha/2 + \alpha/2) = f(\alpha/2) f(\alpha/2) = (f(\alpha/2))^2$

$p \in \mathbb{Q}^*$

$p = f(r) = f(r/2 + r/2) = f(r/2) f(r/2) = (f(r/2))^2$

$\Rightarrow f(r/2) = \sqrt{p}$ but $\sqrt{p}$ can be irrational as p is arbitrary

$r/2 \in \mathbb{Q}$ So $\Rightarrow\Leftarrow$

So f is isomorphism

So $\mathbb{Q} \not\cong \mathbb{Q}^*$

Q> G is group of order p^2 (p is prime) then G is abelian

Ans:- G is either cyclic or $\mathbb{Z}_p \times \mathbb{Z}_p$ form

abelion

$(a, b), (c, d) \in \mathbb{Z}_p \times \mathbb{Z}_p$

1st coordinate order p So abelian

2nd " " " " "

So whole abelion

$(a,b)(c,d)$

$= (ac, bd) = (ca, db) = (c,d)(a,b)$

Q> G is of order p^3. Then either G is abelion or $|Z(G)| = p$

Ans:- $Z(G) \leq G$

$|Z(G)| \mid |G| = p^3$

$|Z(G)| = 1, p, p^2, p^3 \to$ done

done done then

$|G/Z(G)| = p$

$G/Z(G)$ is cyclic

$\Rightarrow G$ is abelion